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AFFINE AND RIEMANNIAN s-MANIFOLDS

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1. Introduction

Let M be a connected Riemmannian manifold, and $I(M)$ the group of all isometries on M. An isometry on M with an isolated fixed point x will be called a *symmetry* at x, and will usually be written as s_x. A point x is an isolated fixed point of a symmetry s_x if and only if s_x induces on the tangent space M_x at x an orthogonal transformation $S_x = (ds_x)_x$ which has no invariant vector. M will be called an *s-manifold* if for each $x \in M$ there is a symmetry s_x at x.

An important case arises when each s_x has order 2. Then M is a symmetric space and $I(M)$ is transitive. Indeed, s_x is the geodesic symmetry at x and the set of all such geodesic symmetries is transitive. It will be shown that the transitivity of $I(M)$ is an implication of the existence of a symmetry s_x at each point x without the assumption of s_x being involutive. Thus we have

Theorem 1 (*F. Brickell*). *If M is a Riemannian s-manifold, then $I(M)$ is transitive.*

The assignment of a symmetry s_x at each point x can be viewed as a mapping $s: M \to I(M)$, and $I(M)$ can be topologised so that it is a Lie transformation group [1]. In this theorem, however, no further assumption on s is made; even continuity is not assumed.

A symmetry s_x will be called a *symmetry of order k* at x if there exists a positive integer k such that $s_x^k = Id.$, and a Riemannian s-manifold with a symmetry of order k at each point will be called a *Riemannian s-manifold* of order[1] k. Clearly a Riemannian s-manifold of order 2 is a symmetric space in the ordinary sense.

Let M be a connected manifold with an affine connection, and $A(M)$ the Lie transformation group of all affine transformations of M. An affine transformation s_x will be called an *affine symmetry* at a point x if x is an isolated fixed point of s_x. The proof of Theorem 1 does not extend to a manifold with affine symmetries. However, assuming differentiability of the mapping $s: M \to A(M)$, we obtain a similar result. A connected manifold with an affine con-

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[1] The concepts of a Riemannian s-manifold and a Riemannian s-manifold of order k were introduced in [2] for the case when the map $s: M \to I(M)$ is differentiable.

nection will be called an *affine s-manifold* if there is a differentiable mapping $s: M \to A(M)$ such that, for each $x \in M$, s_x is an affine symmetry at x.

Theorem 2. *If M is an affine s-maifold, then $A(M)$ is transitive.*

The proof of Theorem 1 is given in § 2. In § 3 Theorem 2 is proved, and in § 4 we describe a class of Riemannian s-manifolds of order k, which are not symmetric spaces. Finally, in § 5 some miscellaneous remarks are made, the differentiability[2] of s usually being assumed.

2. Proof of Theorem 1

We first prove a lemma for later use.

Lemma. *Let G be a topological transformation group acting on a connected topological space M. If, for each point x in M, the G-orbit of x contains a neighborhood of x, then G is transitive on M.*

This assumption will be referred to as local transitivity of G at a point x.

Proof. Since G is transitive on each orbit, for each x the G-orbit $G(x)$ of x is open by our assumption. The complement $C(x)$ of $G(x)$ in M is also open, being a union of orbits. Thus $G(x)$ is open and closed. It is non-empty and therefore coincides with the connected space M. Thus G is transitive.

Proof of Theorem 1. To simplify notation we write $I(M) = G$. Let x be any point in M, and U a normal neighbourhood of x with radius a. Let y be any point in U and let $b = d(x, y)$, the distance between x and y. Let r be the distance from x to the G-orbit $G(y)$ of y; thus

$$r = \operatorname*{Inf}_{f \in G} d(x, f(y)) .$$

Clearly we have $r \leq b < a$, since $y \in G(y)$. Hence there exists a sequence (y_n) in $G(y)$ such that $d(x, y_n) \leq b$, $\lim_{n \to \infty} d(x, y_n) = r$, and the sequence (y_n) converges to a point z in the closed ball with centre x and radius b. Since M is a connected locally compact metric space, orbits are closed. Hence $z \in G(y)$ and $d(x, z) = r$.

Suppose r is positive. Then there exists a unique geodesic segment joining x and z with length $r > 0$. Let w be any point on this geodesic between x and z, and consider the effect of the symmetry s_w at w on z. Clearly $s_w(z)$ belongs to $G(y)$ and is different from z. Since the points x, z, w and $s_w(z)$ are all in U, and the triangle inequality holds for any geodesic triangles in U, we have

$$\begin{aligned} d(x, s_w(z)) &< d(x, w) + d(w, s_w(z)) \\ &= d(x, w) + d(w, z) \\ &= d(x, z) = r , \end{aligned}$$

[2] "Differentiable" will mean "differentiable of class C^∞".

which contradicts the fact that $r = d(x, G(y))$. Thus we have $r = 0$, and hence $x \in G(y)$. Consequently $y \in G(x)$, and since y is an arbitrary point in U we have $U \subset G(x)$. Then by the above lemma, G is transitive on M.

3. Proof of Theorem 2

Put $G = A(M)$. We choose a normal neighbourhood U with origin o which is a normal neighbourhood of each of its points. Then since $A(M)$ is a transformation group on M and the map $s: M \to A(M)$ is continuous it follows that there is a neigobourhood $V \subset U$ sufficiently small that $s_x(o) \in U$ for all x in V. Since U is a normal neighbourhood as above, Exp_x^{-1} is deffned on U for all x in U. Since s_x is an affine transformation, it follows that if $x \in V$ then

$$(1) \qquad s_x(o) = \mathrm{Exp}_x S_x \mathrm{Exp}_x^{-1}(o) ,$$

where S_x is the differential of s_x at x. We note that S_x is a non-singular linear transformation on the tangent space M_x of M at x with no eigenvalue equal to 1. We then have a mapping $h: V \to U$ defined by $h(x) = s_x(o)$ for any x in V. Since the mapping $s: M \to A(M)$ is differentiable, so is h. From the expression (1) for $s_x(o)$ the differential dh_0 of h at the point o is given by $dh_0 = I - S_0$, which is non-singular because no eigenvalue of S_0 is equal to 1. Hence h is a diffeomorphism on some neighbourhood $W \subset U$ of o, and $h(W)$ is a neighbourhood of o contained in the G-orbit $G(o)$ of o. Therefore, by the lemma in § 2, $A(M)$ is transitive.

4. A class of s-manifolds of order k

Let G be a compact connected Lie group, and G^* the diagonal of $G \times G$. Then it is well known that $(G \times G)/G^*$ is a symmetric space and is diffeomorphic to G. We now consider the more general case of G^{k+1}/G^* where G^{k+1} is the direct product of G with itself $k+1$ times, and G^* is the diagonal of G^{k+1}. The coset space G^{k+1}/G^* is then diffeomorphic to G^k under the mapping

$$(x_1, \cdots, x_{k+1}) G^* \to (x_1 x_{k+1}^{-1}, \cdots, x_k x_{k+1}^{-1}) ,$$

and the corresponding action of G^{k+1} on G^k is given by

$$(x_1, \cdots, x_{k+1})(y_1, \cdots, y_k) = (x_1 y_1 x_{k+1}^{-1}, \cdots, x_k y_k x_{k+1}^{-1}) .$$

It follows that G^{k+1} is a transitive transformation group on G^k with G^* as isotropy group at the identity of G^k. For any point $(x_1, \cdots, x_k)$ in G^k we will identify the tangent space with $G_{x_1} \oplus \cdots \oplus G_{x_k}$ by means of the standard projections π_i, $i = 1, \cdots, k$, of G^k onto G. In particular, we write $x^{(i)}_{(x_1,\cdots,x_k)}$ for the vector at $(x_1, \cdots, x_k)$ such that $\pi_i X^{(i)}_{(x_1,\cdots,x_k)} = X_{x_i}$, $\pi_j X^{(i)}_{(x_1,\cdots,x_k)} = 0$ for $i \neq j$. We also write $Ad(x, \cdots, x)$ for the differential of any element $(x,$

$\cdots, x) \in G^*$ evaluated at the identity of G^k. Thus for $X_1, \cdots, X_k \in G_e$ we have

$$Ad(x, \cdots, x)(X_1, \cdots, X_k) = (Ad(x)X_1, \cdots, Ad(x)X_k) .$$

A Riemannian structure on G^k is G^{k+1}-invariant if and only if it is induced from an $Ad(G^*)$-invariant positive definite bilinear form B at the identity of G^k. We write

$$B_{ij}(X, Y) = B(X^{(i)}, Y^{(j)}) .$$

Then B is $Ad(G^*)$-invariant if and only if each B_{ij} is $Ad(G)$-invariant. Since G is compact, it follows that $Ad(G)$ is also compact, and hence on G_e there exists a positive definite bilinear form ϕ invariant under $Ad(G)$. We may choose such a form for each B_{ij} and hence obtain B at the identity of G^k. Then an invariant quadratic form on G^k is obtained by left translation.

Consider the mapping $\sigma: G^{k+1} \to G^{k+1}$ defined by

$$p_1 \circ \sigma = p_{k+1} ,$$
$$p_i \circ \sigma = p_{i-1} \qquad \text{for } i = 2, \cdots, k+1 ,$$

where $p_1, \cdots, p_{k+1}$ are the projections of G^{k+1} onto its factors. Clearly σ is an automorphism of G^{k+1} such that $\sigma^{k+1} = Id$. Let $\pi: G^{k+1} \to G^k$ be the projection defined by

$$(2) \qquad (\pi_i \circ \pi)(x_1, \cdots, x_{k+1}) = x_i x_{k+1}^{-1} , \qquad i = 1, \cdots, k .$$

Then the map $s: G^k \to G^k$ defined by

$$(3) \qquad s \circ \pi = \pi \circ \sigma$$

has the identity of G^k as an isolated fixed point and $s^{k+1} = Id$. We now seek a G^{k+1}-invariant Riemannian structure B on G^k for which s is a symmetry of order $k+1$. It follows from (2) and (3) that at the identity of G^k,

$$(4) \qquad ds\, X^{(i)} = X^{(i+1)} , \qquad i \neq k ,$$

$$(5) \qquad ds\, X^{(k)} = -(X^{(1)} + \cdots + X^{(k)}) .$$

Hence s is a symmetry of order $k+1$ if and only if for $1 \leq i, j \leq k-1$, and $X, Y \in G_e$,

$$(6) \qquad B(X^{(i)}, Y^{(j)}) = B(X^{(i+1)}, Y^{(j+1)}) ,$$

$$(7) \qquad B(X^{(i)}, Y^{(k)}) = -B(X^{(i+1)}, Y^{(1)} + \cdots + Y^{(k)}) ,$$

$$(8) \qquad B(X^{(k)}, Y^{(k)}) = B(X^{(1)} + \cdots + X^{(k)}, Y^{(1)} + \cdots + Y^{(k)}) .$$

From (6) and (7) we have for $1 \leq i \leq k-2$

$$B(X^{(i+2)}, Y^{(1)} + \cdots + Y^{(k)}) + B(X^{(i+1)}, Y^{(k)}) - B(X^{(i+2)}, Y^{(1)}) + B(X^{(i)}, Y^{(k)}) = 0.$$

The first two terms of this equation are zero by (7), and hence

$$(9) \qquad B(X^{(i)}, Y^{(k)}) = B(X^{(i+2)}, Y^{(1)}) .$$

We note that (8) is a consequence of (6) and (7), for (6) implies

$$B(X^{(1)}, Y^{(1)} + \cdots + Y^{(k)}) = B(X^{(1)} + \cdots + X^{(k)}, Y^{(k)}) .$$

Hence, using (7),

$$B(X^{(1)} + \cdots + X^{(k)}, Y^{(1)} + \cdots + Y^{(k)}) = B(X^{(1)} + \cdots + X^{(k)}, Y^{(k)}) - B(X^{(1)}, Y^{(k)}) - \cdots - B(X^{(k-1)}, Y^{(k)}) = B(X^{(k)}, Y^{(k)}) .$$

It follows that (6), (7) and (8) are equivalent to

$$(10) \qquad B_{ij} = B_{i+1,j+1} , \qquad 1 \leq i, j \leq k-1 ,$$

$$(11) \qquad B_{ik} = B_{1,i+2} , \qquad 1 \leq i \leq k-2 ,$$

$$(12) \qquad B_{11} + 2B_{12} + B_{13} + B_{14} + \cdots + B_{1k} = 0 ,$$

where (12) is obtained from (7) with $i = 1$. By means of (10) and (11) we can reduce (12) to

$$B_{11} + 2(B_{12} + \cdots + B_{1\,\frac{k}{2}+1}) = 0$$

for even k, and

$$B_{11} + 2(B_{12} + \cdots + B_{1\,\frac{k+1}{2}}) + B_{1\,\frac{k+3}{2}} = 0$$

for odd $k > 1$.

The system of equations (10), (11) and (12) has the (not necessarily unique) solution

$$B_{ii} = k\phi ,$$

$$B_{ij} = -\phi \qquad \text{for } i \neq j ,$$

where ϕ is a positive definite quadratic form on G_e invariant under $A(G)$. We then have

$$B((X_1, \cdots, X_k), (X_1, \cdots, X_k)) = k \sum_{i=1}^{k} \phi(X_i, X_i) - 2 \sum_{i<j} \phi(X_i, X_j) = \sum_{i=1}^{k} \phi(X_i, X_i) + \sum_{i<j} \phi((X_i - X_j), (X_i - X_j)) .$$

Clearly B is positive definite. By means of left translation by G^k we obtain a Riemannian structure, also written as B, on G^k.

We now prove that G^k together with the Riemannian structure B is not locally symmetric and hence not symmetric. Thus let ∇ be the affine connection and R the curvature tensor field associated with B. We show that $\nabla R \neq 0$ at the identity of G^k. The connection ∇ can be determined by noting that if X is a left invariant vector field on G then, for $1 \leq i \leq k$, $X^{(i)}$ is a left invariant vector field on G^k. Hence, for $1 \leq i, j \leq k$, $B(X^{(i)}, Y^{(j)})$ is a constant. Let $\{X_\alpha\}$, $\alpha = 1, \cdots, r$, be a basis for the vector space of left invariant vector fields on G, which is orthonormal with respect to ϕ. Then $\{X_\alpha^{(i)}\}$, $\alpha = 1, \cdots, r$, $i = 1, \cdots, k$, is a basis for left invariant vector fields on G^k, and it follows easily from the above remark that

$$\begin{aligned} B(\nabla_{X_\alpha^{(i)}} X_\beta^{(j)}, X_\gamma^{(p)}) = \tfrac{1}{2}\{&B([X_\alpha^{(i)}, X_\beta^{(j)}], X_\gamma^{(p)}) + B([X_\gamma^{(p)}, X_\alpha^{(i)}], X_\beta^{(j)}) \\ &+ B([X_\gamma^{(p)}, X_\beta^{(j)}], X_\alpha^{(i)})\}\,. \end{aligned} \tag{13}$$

The connection ∇ is completely determined by (13), and it follows that if X, Y are left invariant vector fields on G then

$$\begin{aligned} &\nabla_{X^{(i)}} Y^{(j)} = \frac{1}{2(k+1)}([X, Y]^{(j)} - [X, Y]^{(i)}) \qquad \text{for } i \neq j\,, \\ &\nabla_{X^{(i)}} Y^{(i)} = \tfrac{1}{2}[X, Y]^{(i)} \qquad \text{not summed for } i. \end{aligned} \tag{14}$$

A straightforward calculation then gives, for $i \neq j$,

$$(\nabla_{X^{(i)}} R)(X^i, X^j)Y^j = \frac{1}{8(k+1)^3}[(2-k^2)((ad\,X)^3 Y)^{(i)} + k((ad\,X)^3 Y)^{(j)}]\,.$$

Thus, for $r > 1$, $\nabla R = 0$ implies that the Lie algebra of G is nilpotent and hence abelian, since G is compact. Hence if G is a compact connected non-abelian Lie group then G^k admits a Riemannian metric, for which it is an s-manifold of order $k+1$, but is not symmetric.

One might also remark[3] that an invariant metric on G^{k+1}/G^* is Riemannian symmetric if and only if it comes from a bi-invariant metric on G^{k+1}. Then it is σ-stable if and only if it has the same projection on each of the $k+1$ factors G of G^{k+1}. Now if $k > 1$ then the group generated by G^* and σ on the tangent space to the identity coset of G^{k+1}/G^* is not irreducible, and it follows immediately that there are many non-locally symmetric Riemannian metrics on G^{k+1}/G^*.

We note that this example and many others are discussed in [4].

[3] The authors wish to thanks the referee for this suggestion as well as other helpful criticisms and comments.

5. Miscellaneous remarks

A) Let M be an affine s-manifold. Since $s:M \to A(M)$ is assumed to be differentiable, the tensor field S of type (1,1) defined by $S_x = ds_x$ at x is differentiable.

We now show that if S is parallel, i.e. $\nabla S = 0$, then the curvature tensor K and the torsion tensor T satisfy $\nabla K = 0$ and $\nabla T = 0$. Therefore the affine connection on M is invariant under parallelism [3].

In fact, let M_x and M_x^* be respectively the tangent and cotangent spaces at x. Take any vectors X, Y, Z in M_x and ω in M_x^*. By parallel translation along each geodesic through x they are extended to local vector fields with vanishing convariant derivative at x.

The torsion tensor T defines a real-valued multilinear function $T_x: M_x^* \times M_x \times M_x \to R$ at each point. Since T is invariant by any affine transformation, we have, in particular,

$$T_x(\omega, X, Y) = T_x(S_x^*\omega, S_xX, S_xY)\,, \tag{15}$$

where S_x^* denotes the transpose of S_x. The covariant derivative ∇T of T is a tensor field of type (1,3), which is invariant by affine transformations. Thus we have

$$(\nabla T)_x(\omega, X, Y, Z) = (\nabla T)_x(S_x^*\omega, S_xX, S_xY, S_xZ)\,. \tag{16}$$

By differentiating (15) covariantly in the direction of S_xZ at x and using (16) we obtain

$$\begin{aligned}(\nabla T)_x(\omega, X, Y, S_xZ) &= (\nabla T)_x(S_x^*\omega, S_xX, S_xY, S_xZ)\\ &= (\nabla T)_x(\omega, X, Y, Z)\end{aligned}$$

Thus $(\nabla T)_x(\omega, X, Y, (I - S_x)Z) = 0$ for any $\omega \in M_x^*$, $X, Y, Z \in M_x$. Since $I - S_x$ is non-singular, we have $(\nabla T)_x = 0$; this holds at all points in M and hence $\nabla T = 0$.

In exactly the same manner we obtain $\nabla K = 0$.

B) If a manifold M with a torsion free connection is an affine s-manifold and has the property as in A), then M is locally symmetric.

C) Let M be a Riemannian s-manifold of order $k > 1$. Assume moreover that the mapping $s:M \to I(M)$ is differentiable. Then the tensor field S defined as in A) satisfies the equation $S^k = I$. The eigenvalues of S are thus k-th roots of 1. It follows from the continuity of S that each root must be constant over M. Since S is real, eigenvalues appear as pairs of conjugates except for the eigenvalue -1, if it exists. At each point x in M we then have the unique eigenspace-decomposition of M_x:

$$M_x = M_{x,-1} \oplus M_{x,1} \oplus \cdots \oplus M_{x,r}, \tag{17}$$

where $M_{x,-1}$ is the eigenspace corresponding to the eigenvalue -1 and $M_{x,i}$, $1 \leq i \leq r$, are the eigenspaces corresponding to the eigenvalues $\cos\phi_i \pm \sin\phi_i \sqrt{-1}$. We thus obtain mutually orthogonal differentiable distributions M_{-1}, M_i, $1 \leq i \leq r$, on M. Corresponding to the decomposition (17) the tensor field S is decomposed into the form

$$S = S_{-1} \oplus S_1 \oplus \cdots \oplus S_r,$$

where each factor acts on the corresponding space in (17). On M_i, $1 \leq i \leq r$, we put

$$F_i = (S_i - I\cos\phi_i)/\sin\phi_i,$$

which is well-defined for each i since $\sin\phi_i \neq 0$. Thus we have a tensor field F of type (1,1) defined by

$$F = 0_{-1} \oplus F_1 \oplus \cdots \oplus F_r$$

where 0_{-1} is the zero tensor on M_{-1}. Obviously F satisfies the equation $F^3 + F = 0$ and has rank equal to $\dim M_1 + \cdots + \dim M_r$.

If S has no real eigenvalue, then $M_{-1} = (0)$ and F is an almost complex structure on M. In addition, F is orthogonal with respect to the Riemannian metric, and hence the metric is almost Hermitian with respect to F. If k is odd, then there is no real eigenvalue. Thus we have

If the mapping $s: M \to I(M)$ is differentiable and has odd order on a Riemannian s-manifold M, then there is an almost complex structure F naturally associated with the given symmetry, and the Riemannian metric is almost Hermitian with respect to F.

D) Let M be a Riemannian s-manifold of order k such that the only eigenvalues of the tensor field S are θ and $\bar{\theta}$ (θ not real). Then either M is a locally symmetric space or $k = 3$.

Proof. At each point $x \in M$ we denote the θ-eigenspace of S_x on the complex tangent space M_x^C by N_x. Then its complex conjugate $\bar{N}_x$ is the $\bar{\theta}$-eigenspace. Let D be the complex distribution which assigns N_x to x, so its complex conjugate $\bar{D}$ is the distribution assigning $\bar{N}_x$ to x. If X is a tangent vector field we write $X \in D$ (resp. $X \in \bar{D}$) to mean that X is tangent to D (resp. $\bar{D}$). If X and Y are complex-valued vector fields, then

$$S_x[X, Y]_x = ds_x[X, Y]_x = [ds\,X, ds\,Y]_x = [SX, SY]_x$$

$$= \begin{cases} (\text{if } X, Y \in D)\ [\theta X, \theta Y]_x = \theta^2[X, Y]_x, \text{ so either } \theta^2 = \bar{\theta} \text{ or } [X, Y] = 0; \\ (\text{if } X, Y \in \bar{D})\ [\bar{\theta} X, \bar{\theta} Y]_x = \bar{\theta}^2[X, Y]_x, \text{ so either } \bar{\theta}^2 = \theta \text{ or } [X, Y] = 0; \\ (\text{if } X \in D, Y \in \bar{D})\ [\theta X, \bar{\theta} Y]_x = [X, Y]_x, \text{ so } [X, Y] = 0. \end{cases}$$

Now write M as a coset space G/K with $G = I(M)$, and K the isotropy subgroup at a point x_0. Then M is a reductive coset space, so the Lie algebra g of G satisfies $g = k + m$ for some $Ad_G(K)$-stable complement m to k in g. If $k \neq 3$, i.e. $\theta^2 \neq \bar{\theta}$ and $\bar{\theta}^2 \neq \theta$, then the above calculation shows that $[m^C, m^C]$ is contained in k^C, so $[m, m]$ is in k, proving that M is locally symmetric.

Suppose furthermore that M is Kaehlerian with respect to the complex structure F given by $F = (S - I\cos\phi)/\sin\phi$, where $\theta = \cos\phi + \sin\phi\sqrt{-1}$. Then F has vanishing covariant derivative, and so does the tensor field $S = I\cos\phi + F\sin\phi$ because $\cos\phi$ and $\sin\phi$ are both constant. By Remark A) M is hence locally symmetric for any k.

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